Section 4.2: Kernel and Image

- Let $f : V \to W$ be a linear transformation from a vector space V to a vector space W, and let $\mathbf{0}_W$ be the zero vector of W. We define the **kernel** of f, denoted $\ker(f)$, to be the set

$$\ker(f) = \{\mathbf{x} \in V \mid f(\mathbf{x}) = \mathbf{0}_W\}.$$

- **Theorem 4.9:** Let $f : V \to W$ be a linear transformation from a vector space V to a vector space W.
  1. Let $\mathbf{0}_V$ be the zero vector of V and let $\mathbf{0}_W$ be the zero vector of W. Then $f(\mathbf{0}_V) = \mathbf{0}_W$.
  2. $\ker(f)$ is a subspace of V.
  3. $\operatorname{im}(f)$ is a subspace of W.
  4. If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation defined as $f(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$ by a matrix $A \in M_{m,n}(\mathbb{R})$, then

$$\ker(f) = \mathrm{NS}(A) \qquad \text{and} \qquad \operatorname{im}(f) = \mathrm{CS}(A).$$

Properties of Linear Transformations

- Let V and W be vector spaces, and let $f : V \to W$ be a linear transformation. We define the **rank** of f to be $\dim(\operatorname{im}(f))$ and the **nullity** of f to be $\dim(\ker(f))$.
- **Theorem 4.13:** Let V and W be vector spaces, and let $f : V \to W$ be a linear transformation.
  1. f is injective if and only if $\ker(f) = \{\mathbf{0}_V\}$.
  2. If $\mathcal{B}$ is a spanning set for V, then $f(\mathcal{B})$ spans $\operatorname{im}(f)$.
  3. If f is injective and $\mathcal{B}$ is a linearly independent subset of V, then $f(\mathcal{B})$ is linearly independent.
  4. If f is injective, then $\operatorname{rank}(f) = \dim V$.
  5. If $\dim V < \infty$, then $\dim V = \operatorname{rank}(f) + \operatorname{null}(f)$.

Properties of Linear Transformations

- **Theorem 4.14:** Let V, W, and Z be vector spaces, and let $f : V \to W$ and $g : W \to Z$ be linear transformations. Then $g \circ f : V \to Z$ is a linear transformation.
- **Theorem 4.15:** Let V and W be vector spaces, and let $f : V \to W$ be a bijective linear transformation. Then the inverse function $f^{-1} : W \to V$ is a linear transformation.
- Let V and W be vector spaces, and $f : V \to W$ a bijective linear transformation. Then f is called an **isomorphism** of vector spaces between V and W, and V and W are said to be **isomorphic**. We write $V \cong W$.

Properties of Linear Transformations

Theorem 4.17: Let V and W be vector spaces, and let $\mathcal{B} = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ be a basis of V. Let $\mathbf{y}_1, \ldots, \mathbf{y}_n \in W$. There is a unique linear transformation $f : V \to W$ such that $f(\mathbf{x}_i) = \mathbf{y}_i$ for all i. Moreover,

1. f is injective if and only if $\{\mathbf{y}_1, \ldots, \mathbf{y}_n\}$ is linearly independent.
2. f is surjective if and only if $\{\mathbf{y}_1, \ldots, \mathbf{y}_n\}$ spans W.
3. f is an isomorphism if and only if $\{\mathbf{y}_1, \ldots, \mathbf{y}_n\}$ is a basis of W.

Corollary 4.18: Let V and W be finite dimensional vector spaces. Then $V \cong W$ if and only if $\dim V = \dim W$.
In particular, every n-dimensional vector space is isomorphic to $\mathbb{R}^n$.